

On the Semisimplicity of the Action of the Frobenius on Etale Cohomology

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Abstract

We give a proof of the semisimplicity of the action of the geometric Frobenius on etale cohomology. The proof is based on [MGM11] and on the Weil Conjectures, ie on the Riemann Hypothesis for non singular projective varieties over finite fields.

Keywords Local Spectra. Algebraic and Topological K Theory l-adic Completion of Spectra.

Introduction:

We are going to work with the abelian category $\mathcal{B}(l)_*$. See the definition below and also [B83]. In [B83] Bousfield defined an universal functor $\mathcal{U}: Z_{(l)} - \text{modules} \mapsto \mathcal{B}(l)_*$ which will be crucial here. We start with the isomorphism ([T89])

$$(1) \quad \pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q \simeq \bigoplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_{(l)}(i))$$

Here $L_{E(1)}$ is the Bousfield localization of $E(1)$ where $E(1)$ is such that $\mathcal{K}_{(l)} = \bigvee_{i=1}^{l-2} \Sigma^{2i} E(1)$. (See below the definition of $E(1)$ and also [B83]) $\mathcal{K}_{(l)}$ is the l localized topological K spectrum \mathcal{K} . In (1), $K(X_\infty)$ is the Quillen's algebraic K theory spectrum on $X_\infty = X \otimes F$, where F is the algebraic closure of the finite field F_q with $q = p^s$ and $p \neq l$. X is a nonsingular projective variety over F_q with finite dimension and d is the dimension of X_∞ . $L_{E(1)}$ denotes $E(1)$ localization, and $[L_{E(1)}K(X_\infty)]^l$ denotes the l -adic completion of the spectra $L_{E(1)}K(X_\infty)$. Here $H_{et}^{2i}(X_\infty, Q_{(l)}(i))$ are the etale cohomology groups with coefficients in Q_l , which are Q_l vector spaces.

We show that $\mathcal{U}(\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q) \simeq [E_0(1)(K(X_\infty))]^l \otimes Q$ (2)

where on the right hand side of the equality, $[E_0(1)(K(X_\infty))]^l$ is the l -adic completion of the $Z_{(l)}$ -module $E_0(1)(K(X_\infty))$ which is an object in \mathcal{B} (See the definition of \mathcal{B} below). This isomorphism is dependent on highly non trivial facts, such as the fact that $\pi_0 L_{E(1)} K(X_\infty)$ is an l -reduced group, which follows from [MGM11]. Being l -reduced group means:

$$\text{Hom}(Q/Z_{(l)}, \pi_0(L_{E(1)} K(X_\infty))) = 0$$

We then obtain the isomorphism: $[(E_0(1)K(X_\infty))]^l \otimes Q \simeq \mathcal{U}(\oplus_{i=0}^{2d} H_{et}(X_\infty, Q_{(l)}(i)))$. Therefore the functor \mathcal{U} allows to study the action of the geometric frobenius $\Phi_X : X \mapsto X$ on $\oplus_{i=0}^{2d} H_{et}(X_\infty, Q_{(l)}(i))$ through the action of $\mathcal{U}(\Phi_X)$ on $[E_0(1)(K(X_\infty))]^l \otimes Q$. We prove that $(E_0(1)(K(X_\infty))) \otimes Q$ is dense in $[E_0(1)(K(X_\infty))]^l \otimes Q$ with the l -adic topology and finally show that $\mathcal{U}(\Phi_X)$ is an Adams operation on $[E_0(1)(K(X_\infty))]^l \otimes Q$ because of the Weil Conjectures and we conclude that the action of Φ_X on the etale cohomological spaces $H_{et}^{2i}(X_\infty, Q_{(l)}(i))$ is semisimple.

The Category $\mathcal{B}(l)_$:*

We begin by describing an abelian category, denoted $\mathcal{B}(l)_*$, equivalent to the category of $E(1)_* E(1)$ -comodules (see [B83], 10.3) Bousfield describes $\mathcal{B}(l)_*$ as follows: Let l be an odd prime and let \mathcal{B} denote the category of $Z_{(l)}[Z_{(l)}^*]$ -modules for the group ring $Z_l[Z_l^*]$, where Z_l^* are the units in $Z_{(l)}$, with the action by the group ring defined by Adams operations $\Psi^k : M \mapsto M$ which are automorphisms and satisfy the following:

- i) There is an eigenspace decomposition

$$M \otimes Q \cong \bigoplus_{j \in \mathbb{Z}} W_{j(l-1)}$$

such that for all $w \in W_{j(l-1)}$ and $k \in Z_{(l)}$,

$$(\Psi^k \otimes id)w = k^{j(l-1)}w$$

- ii) For all $x \in M$ there is a finitely generated submodule $C(x)$ containing x , satisfying: for all $m \geq 1$ there is an n such that the action of $Z_{(l)}^*$ on

$C(x)/l^m C(x)$ factors through the quotient of $(Z/l^{n+1})^*$ by a subgroup of order $l-1$.

To build the category $\mathcal{B}(l)_*$ out of the above category \mathcal{B} , we additionally need the following:

Let $T^{j(l-1)} : \mathcal{B} \mapsto \mathcal{B}$ with $j \in Z$ denote the following equivalence:

For all M in \mathcal{B} , $T^{j(l-1)}(M) = M$ as $Z_{(l)}$ -module, but not as $Z_{(l)}[Z_{(l)}^*]$ -module since the Adams operations in $T^{j(l-1)}(M)$ are now $k^{j(l-1)}\Psi^k : M \mapsto M$ where Ψ^k is the Adams operation of multiplication by k in \mathcal{B} . Now an object in $\mathcal{B}(l)_*$ is defined as a collection of modules $M = (M_n)_{n \in Z}$, with M_n in \mathcal{B} together with a collection of isomorphisms for all $n \in Z$,

$$T^{l-1}(M_n) \mapsto M_{n+2(l-1)}$$

Note that the category \mathcal{B} can be viewed as the subcategory of $\mathcal{B}(l)_*$ consisting of those objects $(M_n)_{n \in Z}$ such that $M_n = M$ if n is congruent to 0 mod $2(l-1)$ and 0 otherwise

In [B83] Bousfield constructs a functor $\mathcal{U} : \pi_*(E(1) - Mod) \mapsto \mathcal{B}(l)_*$. For $H \in \pi_*(E(1) - Mod)$, let \mathcal{U} in \mathcal{B} consist of the objects $\mathcal{U}(H_n)$ in \mathcal{B} for all $n \in Z$.

The Spectrum $E(1)$ and its homology theory $E(1)_$:*

Given $E(1)$, which by construction depends on the prime l , there is a map $E(1) \mapsto \mathcal{K}_l$ which is a ring morphism (see [R] Chapter VI Theorem 3.28) and verifies the equivalence $\mathcal{K}_{(l)} = \bigvee_{i=1}^{l-2} \Sigma^{2i} E(1)$. There are Adams operations $\Psi^k : E(1) \mapsto E(1)$ with k in Z_l^* which are the units in Z_l . These Adams operations are ring spectra equivalences and Ψ^k carries ν^j to $k^{j(l-1)}\nu^j$ in $\pi_{2j(l-1)} E(1)$ for each integer j where ν is such that $\pi_* E(1) = Z_{(l)}[\nu, \nu^{-1}]$ and ν has degree $2(l-1)$. Another property of $E(1)$ is that $E(1)$ localization is the same as $calK_{(l)}$ localization.

The homology $E(1)_*(X)$ with X a spectrum also has Adams operations $\Psi^k : E(1)_*(X) \mapsto E(1)_*(X)$. One checks that $\Psi^k(\nu^j x) = k^{j(l-1)}\nu^j \Psi^k(x)$ for each integer j and k in Z_l^* and $x \in E(1)_*(X)$. The multiplication by ν^j induces an isomorphism $\nu^j : T^{j(l-1)} E(1)_n(X) \mapsto E(1)_{n+2j(l-1)}(X)$ in $\mathcal{B}(l)_*$ for each $j, n \in Z$. It follows that $E(1)_*(X)$ is in $\mathcal{B}(l)_*$ for each spectrum X in \mathcal{S} by taking $E(1)_n(X) = M_n$ defined in 1.1 and by taking as Adams operations, the Adams operations just mentioned.

Remarks 1.

a) We know from ([B83]page 929) that $\mathcal{U}: \pi_*(E(1) - Mod) \mapsto \mathcal{B}(l)_*$ verifies:

$$\mathcal{U}(G) = E(1)_*E(1) \otimes_{\pi_*E(1)} G$$

for all $\pi_*(E(1))$ -module G . In particular taking 0 component, $\mathcal{U}_0(G) = E_0(1)E(1) \otimes_{Z(l)} G$

Therefore if $\phi: G \mapsto G$ is a map of $Z(l)$ -modules with an eigenvalue λ , then $\mathcal{U}(\phi)$ has also eigenvalue λ in $\mathcal{U}(G)$.

b) The next proposition is also proven in [T89]

Proposition 1: *Under the hypothesis that $\pi_0(L_{E(1)}(K(X_\infty)))$ is l -reduced we get:*

$$\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q \simeq [\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q$$

Proof: Let $G_\nu = \pi_0(Y \wedge M(Z/l^\nu)) = \pi_0(T_\nu) = \pi_0(Y/l^\nu)$, where $Y = L_{E(1)}K(X_\infty)$. There is an exact sequence ([BK72] Chap 9)

(3)

$$0 \mapsto \lim^1(G_\nu) \mapsto \pi_0(\text{hom}\lim T_\nu) \mapsto \lim(G_\nu) \mapsto 0$$

and (4)

$$G_\nu = \pi_0(Y \wedge M(Z/l^\nu)) \simeq (\pi_0(Y) \otimes Z/l^\nu) \oplus \text{Tor}^1(Z/l^\nu, Y)$$

Now $\lim \text{Tor}^1(Z/l^\nu, Y) = 0$ since the limit is equal to

$$\Pi_{i=1}^\infty \{g_i = l^i - \text{torsion} - \text{element} \in \pi_0(L_{E(1)}K(X_\infty))/lg_{i+1} = g_i\}$$

and each coordinate in this limit is 0, for it belongs to the intersection of all $l^i(\pi_0 L_{E(1)}K(X_\infty))$ which is 0 because $\pi_0 L_{E(1)}K(X_\infty)$ is reduced. Then by (4), I get:

(5)

$$\lim(G_\nu) = \lim(\pi_0(Y) \otimes Z/l^\nu) = [\pi_0(Y)]^l$$

Obviously (6): $\lim^1(\pi_0(Y) \otimes Z/l^\nu) = 0$. On the other hand, $\lim^1 \text{Tor}^1(Z/l^\nu, Y)$ has bounded l -torsion. Let me show why this is so:

Let $M_\nu = \text{Tor}^1(Z/l^\nu, Y)$. The map $M_{\nu+1} \mapsto M_\nu$ is the map which goes from the $l^{\nu+1}$ -torsion elements of Y to the l^ν -torsion elements of Y given by $x \mapsto lx$. It is in general not surjective, so that it is difficult to prove that $\lim^1 M_\nu = 0$. Anyway, (3) has simplified because of (5) and (6) to

$$(7) \quad 0 \mapsto \lim^1(\text{Tor}^1(Z/l^\nu, \pi_0(Y))) \mapsto \pi_0(\text{hom}\lim Y \wedge M(Z/l^\nu)) = \pi_0(Y^l) \mapsto \lim(G_\nu) = [\pi_0(Y)]^l \mapsto 0$$

where $Y = L_{E(1)}K(X_\infty)$ and $\lim(G_\nu) = [\pi_0(Y)]^l$ is reduced since it is the projective limit of the reduced groups G_ν . See [MG10]. $[\pi_0(Y)]^l$ is also a cotorsion group (see [F1]), since it is the epimorphic image of a cotorsion group in the exact sequence (7): $\pi_0(Y^l)$ is equal to the cotorsion reduced group $\text{Ext}^1(Q/Z_{(l)}, Y)$ since (see [B79])

$$\text{Ext}^1(Q/Z_{(l)}, Y) \mapsto \pi_0(Y^l) \mapsto \text{Hom}(Q/Z_{(l)}, \pi_{-1}(Y))$$

and $\text{Hom}(Q/Z_{(l)}, \pi_{-1}(Y)) = 0$ since $\pi_{-1}(Y)$ is l -reduced, see ([MG10]) and therefore, $\text{Ext}^1(Q/Z_{(l)}, Y) \simeq \pi_0(Y^l)$ and $\text{Ext}^1(Q/Z_{(l)}, Y)$ is a cotorsion group. The torsion group of the cotorsion reduced group $[\pi_0(Y)]^l$ noted $T([\pi_0(Y)]^l)$ is in the terminology of [F2] an l -complete torsion group. Now $[\pi_0(Y)]^l$ is a reduced algebraically compact group since it is complete in the terminology of [R08] page 440. It is complete because it is the closure in the l -adic topology of the topological Hausdorff group $\pi_0(Y)$. Being reduced and algebraically compact implies it is a direct summand of a direct product of cyclic l -groups by [F2] Corollary 38.2 page 161. Henceforth, $T([\pi_0(Y)]^l)$ is contained in a direct product of cyclic l -groups, and so the torsion part of $\pi_0(Y)$, noted $T(\pi_0(Y))$ is contained in a direct sum of cyclic l -groups. Since $T(\pi_0(Y))$ is reduced, then it has bounded l -torsion, ie there exists ν_0 such that $l^{\nu_0}T(\pi_0(Y)) = 0$. Then by definition of \lim^1 , $\lim^1(\text{Tor}^1(Z/l^\nu, \pi_0(Y)))$ has bounded l -torsion, as wanted. Then, tensoring with Q in the exact sequence (7) becomes,

$$(8) \quad 0 \mapsto \pi_0(Y^l) \otimes Q \mapsto [\pi_0(Y)]^l \otimes Q \mapsto 0$$

and therefore Proposition 1 has been proved.

Remark 2: We conjecture that $\pi_0(Y^l)$ and therefore also $[\pi_0(Y)]^l$ are without torsion, in which case by [R08] page 445 $[\pi_0(Y)]^l$, is a direct summand of copies of Z_l . Since tensored by Q , ie $[\pi_0(Y)]^l \otimes Q$, using (1) and the above Proposition 1, is a finite direct sum of copies of Q_l , $[\pi_0(Y)]^l$ has to be a finite direct sum of copies of Z_l .

$$\text{Theorem 1: } \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q) \simeq [E(1)_0(K(X_\infty))]^l \otimes Q$$

Corollary 2: $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_{(l)}(i))) \simeq [E(1)_0(K(X_\infty))]^l \otimes Q$.

Proof: $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_{(l)}(i))) \simeq \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q)$. This fact follows from Proposition 1 and from (1). Hence, corollary 2 follows immediately from theorem 1.

Proof of theorem 1:

Since $\pi_0(L_{E(1)}K(X_\infty))$ is l -reduced the kernel of the l -completion $\pi_0(L_{E(1)}K(X_\infty)) \mapsto [\pi_0(L_{E(1)}K(X_\infty))]^l$ is equal to 0 and the completion map is injective. Also the l -adic topology in $\pi_0(L_{E(1)}K(X_\infty))$ is Hausdorff and this space is dense in its l -completed space. The functor \mathcal{U} is exact and $0 \mapsto \mathcal{U}(\pi_0(L_{E(1)}K(X_\infty)) \otimes Q) \simeq E(1)_0K(X_\infty) \otimes Q \mapsto \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q)$ has dense image. On the other hand since $\pi_0(L_{E(1)}K(X_\infty))$ is l -reduced, $\mathcal{U}(\pi_0(L_{E(1)}K(X_\infty)) \otimes Q) \simeq E(1)_0K(X_\infty) \otimes Q$ is l -reduced (See [B83]) and the isomorphism holds as an isomorphism of Q -vector spaces. Then,

$$0 \mapsto E(1)_0K(X_\infty) \otimes Q \mapsto [E(1)_0K(X_\infty)]^l \otimes Q$$

with dense image. By uniqueness of the l -completed space, (9): $\mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l) \otimes Q \simeq [E(1)_0(K(X_\infty))]^l \otimes Q$, and the theorem follows.

Remark 3: If the conjecture stated in remark 2 holds, then by (9), remark 2, and remark 1 b), $[E(1)_0(K(X_\infty))]^l$ is a finite direct sum of copies of $Z_l[[t]]$, a fact which was proved for nonsingular complete curves in [DM95].

Theorem 2: *The geometric frobenius Φ_X acts semisimply in $\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l(i))$.*

Before we give a proof we need to state one definition and two remarks:

Definition 1: Let E and F be cohomology theories. A natural transformation from $E^n(-)$ to $F^m(-)$ is called a cohomology operation from $E^n(-)$ to $F^m(-)$. If it is compatible with the suspension isomorphisms then it is called a stable operation. (See [Bo95])

Remark 4: Let E and F be spectra. Then the set of stable cohomology operations from E to F can be identified with $F^*(E)$ [Bo95]. Therefore the ring $E(1)^*E(1)$ may be identified with the stable operations of degree 0, $\phi : E(1)_*(-) \mapsto E(1)_*(-)$ which in turn are induced by map of spectra $\phi : E(1) \mapsto E(1)$ (See [KJ84] page 57) Therefore given a base for $E(1)^*E(1)$, we obtain a base for the stable operations of degree 0 on $E(1)_*(-)$.

Remark 5: From ([CCW05] page 13), we know that $\widehat{E(1)}E(1)$ is isomorphic

to $Z_l[[Y]]$, where $Y = \widehat{\Psi^r - 1}$ for the Adams operation Ψ^r with r a primitive modulo l^2 and where $\widehat{E(1)}$ is the l -adic completion of $E(1)$. It is in particular from this fact that Bousfield obtains in ([B83]page 908) an equivalence between the category $\widehat{\mathcal{B}}(l)$ and the category $\mathcal{B}(l)^r$. This isomorphism gives us a base for the ring $\widehat{E(1)}E(1)$, and by remark 4, it gives us a base for the stable degree 0 operations $\phi : E(1)_*(-) \mapsto E(1)_*(-)$, and in particular for the 0 component degree 0 operations $\phi : E(1)_0(-) \mapsto E(1)_0(-)$

Proof of theorem 2:

By the Weil Conjectures the eigenvalues of Φ_X acting on $H_{et}^{2i}(X_\infty, Q_l(i))$ are algebraic numbers all of whose complex conjugates have absolute value q^i .

Then $\mathcal{U}(\Phi_X)$ acting on $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l(i)))$ has eigenvalues whose complex conjugates have absolute value $q^i, i \in 1, 2, ..d$.

We will prove in a moment that this map can be identified with the Adams operation Ψ^q on $[E(1)_0 K(X_\infty)]^l$ which is an object in \mathcal{B} . Now, this Adams operation, Ψ^q , is diagonalizable on $[E(1)_0 K(X_\infty)]^l \otimes Q$. Then, by corollary 2, it is diagonalizable in $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l))$. This fact in turn implies that $\mathcal{U}(\Phi_X)$ is also diagonalizable in $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l))$, which then implies that Φ_X is diagonalizable in $\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l)$ as wanted.

$\mathcal{U}(\Phi_X)$ can be identified with the Adams operation Ψ^q because by the isomorphism of remark 5, $\mathcal{U}(\Phi_X)$ is an infinite combination of the elements, $(\Psi^r - 1)^s$ $s = 0, 1, 2, 3, ...$. This last fact says that the eigenvalues of $\mathcal{U}(\Phi_X)$ on $[E(1)_0 K(X_\infty)]^l \otimes Q$ are a combination of the of the eigenvalues of $(\Psi^r - 1)^s$, $s = 0, 1, 2, 3, ...$ on the same space, which are real numbers, implying that the eigenvalues of $\mathcal{U}(\Phi_X)$ are real numbers. Since on the other hand they are algebraic numbers whose complex conjugate have absolute value q^i , they must be equal to q^i and hence it is the Adams operation Ψ^q , as stated above.

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